States of Nuclear Quadrupole Vibrations and the Two-Phonon Triplets

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A complete classification of the states of an n-dimensional isotropic harmonic oscillator is explicitly given in terms of a product of phonon creation operators applied to the "vacuum." The states are labeled by the phonon number and a set of integers associated with rotation subgroups of U_n. For the three-dimensional case it corresponds to the Bargmann-Moshinsky solution. For $n=5$ (nuclear quadrupole vibrations), the solution contains Rakavy's seniority. In this case, appropriate linear combinations of the above states are constructed in order to get eigenstates of definite angular momentum. Anharmonic terms of fourth order in the creation and destruction operators breaking down the U_5 symmetry are proposed, in order to account for the existence of the two-phonon triplets. Results of the numerical application for a number of even-even nuclei are given.

I. INTRODUCTION

THE classification of the states of the n-dimensional isotropic harmonic oscillator is possible in a variety of ways. In the present work a complete classification is presented and explicit expressions for the states are given in terms of a product of phonon creation operators applied to a "vacuum." These states are labeled by the phonon number ν and a set of nonnegative integers $k_n, k_{n-1}, \cdots, k_3$, which are associated with the i-dimensional rotation subgroups R_i of U_n in the chain

$$U_n \supset R_n \supset R_{n-1} \supset \cdots \supset R_3 .$$

For $n=3$, we get the Bargmann-Moshinsky solution.[1] The case $n=5$ is relevant in the study of the nuclear quadrupole vibrations[2] and the states have definite seniority (k_5) first introduced by Rakavy.[3] In this case, since we are interested in the spin of the levels, appropriate linear combinations of the above solutions are constructed corresponding to states of given phonon number, definite seniority, and angular momentum. This classification corresponds to the chain

$$U_5 \supset R_5 \supset \mathfrak{D}^{(2)}(R_3) ,$$

where $\mathfrak{D}^{(2)}(R_3)$ is the five-dimensional irreducible representation of the R_3 group.[4]

Finally, in the light of the above group-theoretical considerations, a phenomenological attempt is made to account for the observed quadrupole two-phonon triplets in vibrational nuclei. Anharmonic terms, breaking the unitary symmetry U_5 of the quadrupole oscillator, are proposed. They are of fourth order in the creation and destruction operators and conserve the phonon number. One is chosen to be the Casimir invariant of the R_5 group, the other being the square $\mathfrak{L}^2$ of the angular momentum, constructed from the generators $\mathfrak{L}_q$ of the $\mathfrak{D}^{(2)}(R_3)$ group. In the basis corresponding to the second-mentioned chain, the Hamiltonian including the anharmonic terms is diagonal.

Numerical applications for a number of vibrational even-even nuclei were made by a least-square fitting of the three parameters $\hbar\omega$, α, and β. The competition of the two anharmonic terms provides the right ordering of the spins of the triplets. An anharmonicity parameter $\lambda_L = E_{2L}/E_{12}$ (the ratio of the energy of the two-phonon level E_{2L} to the one-phonon level E_{12}) greater than two is obtainable, in contrast with a previous analysis by Kerman and Shakin,[5] in terms of anharmonic cubic terms.

II. CLASSIFICATION OF THE STATES OF THE N-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

The Hamiltonian of the system is

$$\mathcal{H}_0 = \tfrac{1}{2}(\mathbf{p}^2 + \mathbf{x}^2) , \quad (\hbar = m = \omega = 1) \tag{2.1}$$

where

$$\mathbf{p}^2 = \sum_{j=1}^{n} p_j^2 , \quad p_j = -i\partial_j .$$

As is known[6] the problem has as symmetry group the unitary group in n dimensions U_n. This group is responsible for the existence of the "accidental degeneracy" of the system, the number $r(\nu)$ of degenerate states for a given phonon number being given by

$$r(\nu) = \binom{\nu+n-1}{n-1} . \tag{2.2}$$

By introducing phonon creation and destruction operators

$$\mathbf{a}^\dagger = 2^{-1/2}(\mathbf{x} - i\mathbf{p})$$
$$\mathbf{a} = 2^{-1/2}(\mathbf{x} + i\mathbf{p}) \tag{2.3}$$

which satisfy

$$[a_i, a_j^\dagger] = \delta_{ij} , \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 , \quad (i, j = 1, 2, \cdots, n) \tag{2.4}$$

the Hamiltonian (2.1) becomes

$$\mathcal{H}_0 = \mathbf{a}^\dagger \cdot \mathbf{a} + n/2 . \tag{2.5}$$

[1] V. Bargmann and M. Moshinsky, Nucl. Phys. 18, 697 (1960).
[2] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 26, No. 14 (1952).
[3] G. Rakavy, Nucl. Phys. 4, 289 (1957).
[4] M. Moshinsky *Physics of Many-Particle Systems*, edited by E. Meeron (Gordon and Breach Science Publishers, Inc., New York, 1964).
[5] A. K. Kerman and C. M. Shakin, Phys. Letters 1, 151 (1962).
[6] G. A. Baker, Jr., Phys. Rev. 103, 1119 (1956).

In this section we present a complete classification of the eigenstates of $\mathcal{H}_0$, based on a previous work[7] on the basis of the irreducible representations of rotation groups in n-dimensions. As shown there, these basic functions were taken to be polynomials in $x_1, x_2, \cdots, x_n$, labeled by a set of integers $k_n, k_{n-1}, \cdots, k_3, |m|$ associated with the chain

$$R_n \supset R_{n-1} \supset \cdots \supset R_3 . \tag{2.6}$$

These polynomials $P_{k_n,k_{n-1},\cdots,k_3,m}(x_1,x_2,\cdots,x_n)$ which are known as hyperspherical harmonics, are obtained as solutions of the eigenvalue problem

$$\begin{gathered} \mathcal{G}_1{}^{(q)} P_{k_n,\cdots,k_3,m} = k_q(k_q+q-2) P_{k_n,\cdots,k_3,m} , \\ L_0 P_{k_n,\cdots,k_3,m} = m P_{k_n,\cdots,k_3,m} \\ (q=3,4,\cdots,n) , \end{gathered} \tag{2.7}$$

where $\mathcal{G}_1{}^{(q)}$ is the Casimir invariant of the R_q group,

$$\mathcal{G}_1{}^{(q)} = -\mathbf{x}^2 \nabla_q{}^2 + (\mathbf{x}\cdot\nabla_q)^2 + (q-2)\mathbf{x}\cdot\nabla_q , \tag{2.8}$$

and

$$L_0 = -i(x_1\partial_2 - x_2\partial_1) .$$

In (2.8) the scalar products refer to the Euclidean space of q dimensions.

Their explicit form is

$$P_{k_n,\cdots,k_3,m}(x_1,x_2,\cdots,x_n) = (x_1 \pm i x_2)^{|m|} \times \prod_{j=3}^{n} C_{k_j k_{j-1}} G_{k_j k_{j-1}}(x_j,R_j) , \quad (k_2 \equiv |m|) \tag{2.9}$$

with

$$G_{k_j k_{j-1}}(x_j,R_j) = \sum_{\mu=0}^{[(1/2)(k_j-k_{j-1})]} (-)^{\mu} \frac{\Gamma(k_j-1+j/2-\mu) R_j{}^{2\mu} x_j{}^{k_j-k_{j-1}-2\mu}}{2^{2\mu}\mu!(k_j-k_{j-1}-2\mu)!} \quad \left(R_j{}^2 = \sum_{i=1}^{j} x_i{}^2\right) . \tag{2.10}$$

The set of integers $k_n, k_{n-1}, \cdots, k_3, |m|$ obey the branching rules

$$k_n \geqslant k_{n-1} \geqslant \cdots \geqslant k_3 \geqslant |m| \geqslant 0 . \tag{2.11}$$

We want to point out here that these polynomials in $x_1, x_2, \cdots, x_n$ have the same form when expressed in terms of creation operators $a_i{}^\dagger$ applied to a "vacuum" $|0\rangle$. In fact, if we express $\mathcal{G}_1{}^{(q)}$ and L_0 in terms of creation and destruction operators, through (2.3), we get

$$\begin{gathered} \mathcal{G}_1{}^{(q)} = -\mathbf{a}^{\dagger 2}\mathbf{a}^2 + (\mathbf{a}^\dagger\cdot\mathbf{a})^2 + (q-2)\mathbf{a}^\dagger\cdot\mathbf{a} , \\ L_0 = -i(a_1{}^\dagger a_2 - a_2{}^\dagger a_1) \quad (q=3,4,\cdots,n) . \end{gathered} \tag{2.12}$$

Since the commutation relations (2.4) allow us to interpret the a_i when applied to a polynomial $P(\mathbf{a}^\dagger)|0\rangle$ as $a_i = \partial/\partial a_i{}^\dagger$ we see that the operators $\mathcal{G}_1{}^{(q)}$ and L_0 have the same expression in terms of $x_1, x_2, \cdots, x_n$ or $a_1{}^\dagger, a_2{}^\dagger, \cdots, a_n{}^\dagger$. Therefore, the same occurs with the basic polynomials $P_{k_n,\cdots,k_3,m}$.

The eigenstates of the Hamiltonian (2.1) obey

$$\mathbf{a}^\dagger\cdot\mathbf{a}|\nu k_n\cdots k_3 m\rangle = \nu|\nu k_n\cdots k_3 m\rangle , \tag{2.13}$$

where ν is the phonon number. It is apparent that (2.13) corresponds to the Euler homogeneity condition and so $|\nu k_n\cdots k_3 m\rangle$ are homogeneous polynomials in $\mathbf{a}^\dagger$, of degree ν, applied to $|0\rangle$. As the $P_{k_n,\cdots,k_3,m}$ are homogeneous of degree k_n in the $a_i{}^\dagger$, in order to obtain $|\nu k_n\cdots k_3 m\rangle$ we multiply the $P_{k_n,\cdots,k_3,m}$ by an invariant of $R_n, R_{n-1}, \cdots, R_3$ which has to be homogeneous of degree $\nu - k_n$. So we have

$$|\nu k_n\cdots k_3 m\rangle = N_{jk_n\cdots k_3 m}(\mathbf{a}^{\dagger 2})^s P_{k_n,\cdots,k_3,m}(\mathbf{a}^\dagger)|0\rangle , \tag{2.14}$$

where $\nu = 2s + k_n$, $s = 0, 1, 2 \cdots, [\nu/2]$. To get the normalization constants $N_{jk_n\cdots k_3 m}$ we assume that each of the $P_{k_i,\cdots,m}$ for $i<n$ is normalized to one and we obtain, using (2.9),

$$C_{k_n k_{n-1}} = \left[\frac{2^{k_n-k_{n-1}}(2k_{n-1}+n-3)!(k_n-k_{n-1})!}{\Gamma(k_n-1+n/2)\Gamma(k_{n-1}-1+n/2)(k_n+k_{n-1}+n-3)!}\right]^{1/2} . \tag{2.15}$$

The $P_{k_n,\cdots,m}$ being normalized, it results easily that

$$N_{\nu k_n\cdots k_3 m} \equiv N_{\nu k_n} = \left[\frac{(n+2k_n-2)!!}{(n+k_n+\nu-2)!!(\nu-k_n)!!}\right]^{1/2} . \tag{2.16}$$

In particular, for $n=3$, the states (2.14) coincide, including normalization, with the Bargmann-Moshinsky solution[1]

$$P_{\nu k_3 m}(\mathbf{a}^+)|0\rangle = \left[\frac{4\pi}{(\nu+k_3+1)!!(\nu-k_3)!!}\right]^{1/2} \times (\mathbf{a}^+)^{\nu-k_3}\mathcal{Y}_{k_3 m}(\mathbf{a}^+) > |0\rangle . \tag{2.17}$$

The cases $n=5$ and $n=7$ are related to the nuclear quadrupole and octopole vibrations, respectively. In the next section the quadrupole vibrations are discussed in greater detail.

III. QUADRUPOLE VIBRATIONS

The states (2.14) for this case have a definite seniority (k_5), in the sense of Rakavy.[3] The classification of the states in terms of (ν,k_5,k_4,k_3,m) is complete, i.e., any two degenerate states in $\mathcal{H}_0$ differ at least in one of the numbers k_5, k_4, k_3, and m.[8] However, since we are

[7] J. A. Castilho Alcarás and P. Leal Ferreira (to be published).

[8] Another complete classification of the quadrupole vibrations was obtained by M. Moshinsky, using the techniques of the "canonical polynomials" [M. Moshinsky, Nucl. Phys. 31, 384

interested in the angular momentum of the levels, appropriate linear combinations of the above states will be constructed. In the present case the angular momentum operator in spherical components is defined by[4]

$$\mathcal{L}_q=6^{1/2}\sum_{m,m'=-2}^{2}(2m1q|2m')\eta_{m'}{}^{\dagger}\eta^m \qquad (3.1)$$

$$(g=-1,0,1)$$

where $\eta_m{}^{\dagger}$ and η^m are the spherical components of $\mathbf{a}^{\dagger}$ and $\mathbf{a}$. Since the states $|\nu k_n\cdots k_3m\rangle$ are given in Cartesian components of $\mathbf{a}^{\dagger}$, we express the $\eta_m{}^{\dagger}$ and η^m in terms of these components by

$$\begin{aligned}\eta_{\pm1}{}^{\dagger}&=\mp2^{-1/2}(a_1{}^{\dagger}\pm ia_2{}^{\dagger}),\\ \eta_0{}^{\dagger}&=a_3{}^{\dagger},\\ \eta_{\pm2}{}^{\dagger}&=2^{-1/2}(a_4{}^{\dagger}\pm ia_5{}^{\dagger}),\end{aligned} \qquad (3.2)$$

with similar components for the η^m. (For lowering and raising spherical indices the metric tensor $g_{mm'}=(-)^m\times\delta_{m',-m'}$ is used.) The expression (3.1) then becomes

$$\begin{aligned}\mathcal{L}_1&=2^{-1/2}(iD_{41}+D_{42}+D_{15}+iD_{52}+i3^{1/2}D_{13}+3^{1/2}D_{32});\\ \mathcal{L}_0&=2D_{45}+D_{12};\\ \mathcal{L}_{-1}&=2^{-1/2}(iD_{41}+D_{24}+D_{51}+iD_{52}+i3^{1/2}D_{13}+3^{1/2}D_{23});\end{aligned} \qquad (3.3)$$

where

$$D_{jk}=-i(a_j{}^{\dagger}a_k-a_k{}^{\dagger}a_j).$$

The following properties of $\mathcal{L}_q$ are relevant:

$$\begin{aligned}[\mathcal{L}_q,\mathcal{L}_{q'}]&=\sum_{q''}(-)^{q''}\epsilon_{qq'-q''}\mathcal{L}_{q''},\\ [\mathcal{L}_q,\mathcal{H}_0]&=[\mathcal{L}_q,\mathcal{G}_1{}^{(5)}]=0.\end{aligned} \qquad (3.4)$$

It follows from (3.4) that we can construct simultaneous eigenkets of $\mathcal{H}_0$, $\mathcal{G}_1{}^{(5)}$, $\mathcal{L}^2$, and $\mathcal{L}_0$ with eigenvalues $\nu+n/2$, $k_5(k_5+3)$, $L(L+1)$, and M, respectively.[9] These eigenfunctions will be linear combinations of $|\nu k_5k_4k_3m\rangle$,

$$|\nu k_5LM\rangle=\sum_{k_4,k_3,m}(\nu k_5k_4k_3m|\nu k_5LM)|\nu k_5k_4k_3m\rangle. \qquad (3.5)$$

Since we are mainly interested in the low-lying states ($\nu=0, 1, 2, 3$), we restricted ourselves to the calculation of transformation brackets for these cases. The results are summarized in Table I. The determination of the values of L, for given ν and k_5 can be done without the knowledge of (3.5). In fact, we can deduce that

$$\mathcal{L}_0|\nu k_5k_4k_3m\rangle=\sum_{k_4'=k_3}^{k_5}A_{k_4k_4'}|\nu k_5k_4'k_3m\rangle$$

$$=i\left[\frac{(k_5-k_4+1)(k_4+k_3+1)(k_5+k_4+2)(k_4-k_3)}{k_4(k_4+1)}\right]^{1/2}$$

$$\times|\nu k_5k_4-1k_3m\rangle+m|\nu k_5k_4k_3m\rangle$$

$$-i\left[\frac{(k_4-k_3+1)(k_5-k_4)(k_4+k_3+2)(k_5+k_4+3)}{(k_4+1)(k_4+2)}\right]^{1/2}$$

$$\times|\nu k_5k_4+1k_3m\rangle. \qquad (3.6)$$

The eigenvalues M of $\mathcal{L}_0$ are obtained by diagonalization of the matrix $A_{k_4k_4'}$ which may be written as

$$A_{k_4k_4'}=m\delta_{k_4k_4'}+ia_{k_5-k_4+1}(k_5,k_3)\delta_{k_4,k_4'-1}-ia_{k_5-k_4+2}(k_5,k_3)\delta_{k_4,k_4'+1}, \qquad (3.7)$$

where

$$a_{k4}(k_5,k_3)=\left[\frac{(k_5-k_4+1)(k_4+k_3+1)(k_5+k_4+2)(k_4-k_3)}{k_4(k_4+1)}\right]^{1/2}.$$

The eigenvalues of (3.7) may be shown to be

$$M=2(k_5-k_3)+m,\ 2(k_5-k_3-2)+m,\ \cdots,\ -2(k_5-k_3)+m. \qquad (3.8)$$

For each value of k_5 we have a set of M values from which we infer the corresponding L values. In Table II

TABLE I. Eigenkets (3.5) for ν from 0 to 3. Apply the lowering weight operator $\mathcal{L}_-$ to get eigenkets corresponding to other values of M.

ν	k_5	L	$\lvert\nu k_5LL\rangle$
0	0	0	$\lvert 00000\rangle$
1	1	2	$2^{-1/2}[\lvert 11100\rangle+i\lvert 11000\rangle]$
2	0	0	$\lvert 20000\rangle$
2	2	2	$(2/7)^{1/2}[\lvert 22210\rangle+i\lvert 22110\rangle-(3/2)^{1/2}\lvert 22222\rangle]$
2	2	4	$(3/16)^{1/2}\lvert 22200\rangle-(5/16)^{1/2}\lvert 22000\rangle+i2^{-1/2}\lvert 22100\rangle$
3	1	2	$2^{-1/2}[\lvert 31100\rangle-\lvert 31000\rangle]$
3	3	0	$(9/70)^{1/2}[\lvert 33322\rangle+i\lvert 33222\rangle-10^{1/2}(1/15)\lvert 33330\rangle+\lvert 3332\bar{2}\rangle+i\lvert 3322\bar{2}\rangle-10^{1/2}(7/15)\lvert 33310\rangle+(14/9)^{1/2}\lvert 33110\rangle]$
3	3	3	$(3/20)^{1/2}[-2^{1/2}\lvert 33333\rangle+\lvert 33321\rangle+i\lvert 33221\rangle-(5/9)^{1/2}\lvert 3331\bar{1}\rangle-i(4/3)^{1/2}\lvert 3321\bar{1}\rangle+(7/9)^{1/2}\lvert 3311\bar{1}\rangle]$
3	3	4	$44^{-1/2}[(20/3)^{1/2}\lvert 33310\rangle+4i\lvert 33210\rangle-(28/3)^{1/2}\lvert 33110\rangle-6^{1/2}\lvert 33322\rangle-i6^{1/2}\lvert 33222\rangle]$
3	3	6	$32^{-1/2}[2^{1/2}\lvert 33300\rangle-14^{1/2}\lvert 33100\rangle+3i\lvert 33200\rangle-i7^{1/2}\lvert 33000\rangle]$

(1962)]. The classification involves the numbers ν, L, M and two non-negative integers l, n subjected to the following branching rules:

$$\nu-L/2\geqslant 3l+2n\geqslant\nu-L \quad (L\text{ even}),$$
$$\nu-L/2-3/2\geqslant 3l+2n\geqslant\nu-L \quad (L\text{ odd})$$

(private communication).

[9] We are here dealing with the chain

$$U_5\supset R_5\supset\mathfrak{D}^{(2)}(R_3)$$

since the $\mathcal{L}_q$ are the generators of $\mathfrak{D}^{(2)}(R_3)$.[4] The classification is then no more complete, since a quantum number is missing. However, for ν up to 5, the seniority and the angular momentum completely classify the states, as firstly pointed out by Rakavy.[3]

TABLE II. Values of the angular momentum L for seniority k_5 from 0 to 7.

k_5	0	1	2	3	4	5	6	7
L	0	2	2, 4	0, 3, 4, 6	2, 4, 5, 6, 8	2, 4, 5, 6, 7, 8, 10	0, 3, 4, 6^2, 7, 8, 9, 10	2, 4, 5, 6, 7, 8^2, 9, 10, 11, 12, 14

we get the L values for k_5 from 0 to 7. We see that for each value of k_5 from 0 to 5, no repeated values of L appear. From (2.14) it follows that for a given ν, k_5 assumes the values ν, $\nu-2$, $\cdots$, 0 or 2. Then for ν from 0 to 5, k_5 and L are sufficient ro classify the states.[3]

IV. TWO-PHONON TRIPLETS OF QUADRUPOLE VIBRATIONS

In this section a phenomenological attempt is made, based on the previous group theoretical analysis, to account for the observed two-phonon triplets (0^+,2^+,4^+) of quadrupole vibrational even-even nuclei.[10] Assuming that the vibration picture is essentially correct, the existence of separated triplets may be interpreted as a breakdown of the U_5 symmetry of the Hamiltonian $\mathcal{H}_0$ by anharmonic terms. These terms were suggested by the symmetries of the chain $U_5\supset R_5\supset \mathfrak{D}^{(2)}(R_3)$ as a linear combination of the Casimir invariant of the R_5 group $\mathcal{I}_1^{(5)}$ and $\mathcal{L}^2$ whose coefficients are adjustable parameters. These operators are of fourth order in $\mathbf{a}^\dagger$ and $\mathbf{a}$ and conserve the phonon number.

TABLE III. Comparison of theoretical and experimental levels.

Nuclei	Phonon number	Spin	Energy (MeV) Experimental	Energy (MeV) Theoretical	Energy ratio to the first 2 level Experimental	Energy ratio to the first 2 level Theoretical
Ni^{62}	1	2	1.172	1.137	1.00	1.00
	2	0	2.048	2.052	1.75	1.80
	2	2	2.302	2.306	1.96	2.03
	2	4	2.336	2.343	1.99	2.06
Pd^{106}	1	2	0.51	0.59	1.00	1.00
	2	0	1.14	1.14	2.24	1.93
	2	2	1.13	1.12	2.22	1.90
	2	4	1.23	1.21	2.41	2.05
Pd^{108}	1	2	0.43	0.49	1.00	1.00
	2	0	1.03	1.03	2.40	2.10
	2	2	0.94	0.93	2.19	1.90
	2	4	1.03	1.01	2.40	2.06
Cd^{112}	1	2	0.62	0.67	1.00	1.00
	2	0	1.23	1.23	1.98	1.84
	2	2	1.30	1.29	2.10	1.93
	2	4	1.40	1.39	2.26	2.07
Cd^{114}	1	2	0.56	0.61	1.00	1.00
	2	0	1.15	1.14	2.05	1.87
	2	2	1.21	1.20	2.16	1.97
	2	4	1.29	1.28	2.30	2.10

The new Hamiltonian is then

$$\mathcal{H}=\mathcal{H}_0+\alpha\mathcal{I}_1^{(5)}+\beta\mathcal{L}^2. \tag{4.1}$$

It is clear that $\mathcal{H}$ is diagonal in the basis $|\nu k_5 LM\rangle$ constructed in the previous section and its eigenvalues are

$$E_{\nu L}(k_5)=\hbar\omega[\nu+5/2+\alpha k_5(k_5+3)+\beta L(L+1)]. \tag{4.2}$$

It is relevant to notice that the existence of only one of the terms $\mathcal{I}_1^{(5)}$ or $\mathcal{L}^2$ in $\mathcal{H}$ does not give the correct ordering of the triplet levels. In fact, since for $\nu=2$ the level 0^+ has seniority zero and 2^+ and 4^+ have both seniority two, the effect of $\mathcal{I}_1^{(5)}$ alone is to displace equally 2^+ and 4^+, relatively to the level 0^+. On the other hand, the effect of $\mathcal{L}^2$ yields only two possible orderings (0^+, 2^+, 4^+) or (4^+, 2^+, 0^+).

The competition of both terms, however, gives the correct ordering and a reasonable spacing of the levels.

The parameters $\hbar\omega$, α, and β are adjusted by a least-square fitting, using the experimental data on the energies of one and two-phonon levels.[10] The fitting provides small absolute values for α and β. As an example we got in the case of Ni^{62} $\alpha=2.32\times10^{-2}$ and $\beta=0.26\times10^{-2}$. The results are contained in Table III for a number of nuclei. (In the cases of cadmium isotopes the two other states 0^+ and 2^+ which appear are presumably intrinsic excitations coupled with collective degrees of freedom.[11]) The energies of the levels are reasonably closed to the experimental values and the ordering of the levels is reproduced in all cases considered. Besides, the anharmonicity parameter $\lambda_L=E_{2L}/E_{12}$ also is near the observed value and may be greater than 2. In this respect the present phenomenological approach turns out better than that proposed by Kerman and Skakin,[5] based on cubic anharmonic terms. As may be seen from their expressions for the perturbed energy levels, $\lambda_L\leqslant 2$ for $L=2$ and 4, a result which is in disagreement with the experimental data for several nuclei. Concerning the three-phonon quintets[12] the present approach predicts two orderings for the levels 0^+, 3^+, 4^+, and 6^+, namely, (0^+,3^+,4^+,6^+) or (6^+,4^+,3^+,0^+) corresponding to β greater or smaller than zero, respectively, while the position of the level 2^+ depends on the algebraic values of α and β. Unfortunately the experimental data are too scarce for a more detailed analysis.

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